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Field theoretic treatment of an effective action for a model of catalyzed autoamplification

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Reaction-diffusion models can exhibit continuous phase transitions in behaviors, and their dynamics at criticality often exhibit scalings with key parameters that can be characterized by exponents. While models with only a single field that transitions between absorbing and nonabsorbing states are well characterized and typically fall in the directed percolation universality class, the effects of coupling multiple fields remain poorly understood. We recently introduced a model which has three fields: one of which relaxes exponentially, one of which displays critical behavior, and one of which has purely diffusive dynamics but exerts an influence on the critical field [Tchernookov *et al.*, J. Chem. Phys. **130**, 134906 (2009)]. Simulations suggested that this model is in a universality class distinct from other reaction-diffusion systems studied previously. Although the three fields give rise to interesting physics, they complicate analysis of the model with renormalization-group methods. Here, we show how to systematically simplify the action for this model such that analytical expressions for the exponents of this universality class can be obtained by standard means. We expect the approach taken here to be of general applicability in reaction-diffusion systems with coupled order parameters that display qualitatively different behaviors close to criticality.

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I. INTRODUCTION

Many systems far from equilibrium can be described by reaction-diffusion models (see [1–3] and references therein). Because fluctuations that are correlated in time and space can profoundly influence the collective behaviors of interest in such models, mean-field treatments are often inadequate, and it becomes necessary to develop approaches that systematically account for stochastic effects. To this end, field theoretic renormalization-group methods have been introduced and used to show that a broad range of reaction-diffusion models belong to only a handful of classes, for each of which the members conform to a set of universal scaling relations [1,2,4–6]. In particular, many reaction-diffusion models that exhibit phase transitions separating absorbing and nonabsorbing states have been shown to belong to the directed percolation (DP) class [2,7–10]. While single-field models of this nature have been exhaustively categorized [11], the few studies of multiple-field models to date [3,12–15] suggest that coupling between order parameters can give rise to rich physics that remains to be explored.

To this end, we recently introduced a model of catalyzed autoamplification [3]. In the model, a molecule A binds the catalyst D to form a bound complex B that can either dissociate or create another copy of A; molecules of A are destroyed at a constant rate. Schematically,

$$\mathrm{A} + \mathrm{D} \underset{f}{\overset{h}{\rightleftarrows}} \mathrm{B} \overset{g}{\rightarrow} \mathrm{B} + \mathrm{A}$$

$$\mathrm{A} \overset{k}{\rightarrow} \varnothing , \qquad (1)$$

where f, g, h, and k are the rate constants for the indicated reactions, and $\mathrm{A} \to \varnothing$ represents the loss of one copy of A. An important feature of these equations is that they conserve the catalyst: $\rho \equiv C_D + C_B = \text{constant}$, where C_Q is the concentration of species $Q(Q=A,B,D)$.

Qualitatively, as the net rate of producing the self-amplifying factor exceeds that of removing it, the system undergoes a phase transition which is superficially similar to that described above for models in the DP universality class. The exponents that characterize the scaling of this model (as well as DP) at criticality are

$$\xi_\perp \sim |\Delta\rho|^{-\nu_\perp}, \quad \xi_\parallel \sim |\Delta\rho|^{-\nu_\parallel}, \quad \text{and } C_A \sim \Delta\rho^{\beta}. \qquad (2)$$

where $\xi_\perp$ and $\xi_\parallel$ are the correlation length and time, respectively, and $\Delta\rho = \rho - fk/gh$ (see [3] for further discussion of the choice of exponents). While the behavior of our model is like that of DP in the mean field, our stochastic simulations suggested that the exponents for catalyzed autoamplification are distinct from those for DP (even a multiple-field version [12]) and another model, the diffusive epidemic process (also known as the model with pollution) [13–15].

Here, we develop a field theoretic treatment of the model of catalyzed autoamplification. The three coupled fields complicate the analysis, and we show in the Appendix how one can systematically eliminate the "massive" field to obtain an effective action in the same universality class. With this action and the standard diagrammatic procedure [2,6] we calculate the critical exponents governing the divergence of the spatial and temporal correlation lengths to be $\nu_\perp = 1/(2-\epsilon/2)$ and $\nu_\parallel = 1 + O(\epsilon^2)$, respectively, and the exponent which describes the behavior of the average value of the order parameter as a function of the distance to the critical point to be $\beta = 1 - \epsilon/8$. We find good agreement between these results and our earlier simulations [3]. The relation to other models is discussed.

II. FORMULATION OF THE STOCHASTIC FIELD THEORY

Below a critical dimension, stochastic fluctuations affect the behavior of the system near the critical point, and they must be taken into account to calculate the critical exponents accurately. To this end, we discretize the system and denote the number of copies of molecule $Q(Q=A,D,B)$ by $\mathbf{n}_Q$ where each component of the vector corresponds to a different site on the resulting lattice. For notational convenience, we define an operator $\hat{E}_{Q,i}$ such that $\hat{E}_{Q,i}F(\{n_Q\})=F(\ldots,n_i+1,\ldots)$ and its inverse such that $\hat{E}_{Q,i}^{-1}F(\{n_Q\})=F(\ldots,n_i-1,\ldots)$ for an arbitrary function F. Then, the master equation that governs the evolution of the probability of states is

$$\frac{\partial P(\mathbf{n}_A,\mathbf{n}_D,\mathbf{n}_B,t)}{\partial t}=\hat{W}P(\mathbf{n}_A,\mathbf{n}_D,\mathbf{n}_B,t) \tag{3}$$

with

$$\begin{aligned}\hat{W}=D_0\sum_{\langle ij\rangle}\sum_{Q}(\hat{E}_{Q,i}\hat{E}_{Q,j}^{-1}-1)n_{Q,i}\\+\sum_i[gn_{B,i}(\hat{E}_{A,i}^{-1}-1)+k(\hat{E}_{A,i}-1)n_{A,i}]\\+\sum_i[h(\hat{E}_{A,i}\hat{E}_{D,i}\hat{E}_{B,i}^{-1}-1)n_{A,i}n_{D,i}+f(\hat{E}_{A,i}^{-1}\hat{E}_{D,i}^{-1}\hat{E}_{B,i}-1)n_{B,i}]\end{aligned} \tag{4}$$

where $\langle ij\rangle$ denotes nearest neighbors. The first (double) sum accounts for diffusion of molecules, the second sum accounts for production and degradation of A, and the third sum accounts for binding and unbinding. Note that the catalyst is allowed to hop from site to site such that its copy number can vary spatially, and the conservation law mentioned in the Introduction holds only for the sum of populations over the whole lattice.

With a view toward applying the tools of quantum field theory to analysis of this master equation, we convert it into second quantized, and, in turn, coherent-state representations following the standard procedure [2,16,17]. To this end, we first formulate the problem in terms of the ladder operators $\hat{a}_x^\dagger,\hat{a}_x,\hat{d}_x^\dagger,\hat{d}_x,\hat{b}_x^\dagger,\hat{b}_x$ with commutation relations

$$[\hat{p}_x,\hat{q}_y^\dagger]=\delta_{p,q}\delta_{x,y}\quad[\hat{p}_x,\hat{q}_y]=0\quad[\hat{p}_x^\dagger,\hat{q}_y^\dagger]=0, \tag{5}$$

where p and q take on the values a, b, d and x and y label lattice sites. These operators enable us to construct a state vector that describes the system

$$|\Psi\rangle=\sum P(\mathbf{n},\mathbf{m},\mathbf{l})\prod_x\hat{a}_x^{\dagger n_x}\hat{d}_x^{\dagger m_x}\hat{b}_x^{\dagger l_x}|0\rangle, \tag{6}$$

where $|0\rangle$ is the vacuum state ($\hat{p}_x|0\rangle=0$ for all p and x) and the sum runs over all possible vectors $\mathbf{n}$, $\mathbf{m}$, and $\mathbf{l}$ with each component equal to a non-negative integer. We can then write the master equation in the form of a Schrödinger equation with imaginary time,

$$\frac{d}{dt}|\Psi\rangle=H|\Psi\rangle. \tag{7}$$

The "Hamiltonian" is non-Hermitian,

$$H=D_0\sum_p\sum_{\langle xx'\rangle}(\hat{p}_x^\dagger-\hat{p}_{x'}^\dagger)(\hat{p}_x-\hat{p}_{x'})+\sum_x H_x^{(r)}, \tag{8}$$

where $H_x^{(r)}$ is the part of the Hamiltonian due to the chemical reactions at site x,

$$\begin{aligned}H_x^{(r)}=g\hat{b}_x^\dagger(\hat{a}_x^\dagger-1)\hat{b}_x+k(1-\hat{a}_x^\dagger)\hat{a}_x-h\hat{a}_x^\dagger\hat{d}_x^\dagger\hat{a}_x\hat{d}_x+h\hat{b}_x^\dagger\hat{a}_x\hat{d}_x\\-f\hat{b}_x^\dagger\hat{b}_x+f\hat{a}_x^\dagger\hat{d}_x^\dagger\hat{b}_x.\end{aligned} \tag{9}$$

We can switch from the Hamiltonian formalism to one of path integrals through harmonic oscillator coherent states [2,18]. These are defined as

$$|\phi_x\rangle=e^{(\phi_x\hat{a}^\dagger-\phi_x^*\hat{a})}|0\rangle, \tag{10}$$

where ϕ_x is a complex number. We define analogous coherent states for the operators b and d which we label by ξ and η, respectively. These coherent states have the property that

$$\int|\phi_x\rangle\langle\phi_x|\frac{d^2\phi_x}{\pi}=1. \tag{11}$$

Consequently, for the time evolution of every observable, we can break the path into M time slices of length Δt and then insert a complete set of coherent states in between each time slice. Taking the limits $\Delta t\to 0$ and $M\to\infty$, we then find that the time evolution of the average of an observable O follows:

$$\begin{aligned}\langle O(t)\rangle=\int D\phi D\phi^*D\eta D\eta^*D\xi D\xi^*O(\phi,\phi^*,\eta,\eta^*,\xi,\xi^*)\\\times e^{-S(\phi,\phi^*,\eta,\eta^*,\xi,\xi^*)}\end{aligned} \tag{12}$$

where $O(\phi,\phi^*,\eta,\eta^*,\xi,\xi^*)$ is the coherent-state representation of the operator O. Note that we have dropped terms in this equation which correspond to the initial conditions because here we only consider the properties of the stationary state and their associated critical exponents. In [14], van Wijland *et al.* are also interested in the initial behavior of the system, and they calculate exponents for quantities like the critical initial slip which are dependent on the initial conditions. Such an investigation is beyond the scope of our study.

The action (S) can be obtained from the Hamiltonian written in a normal-ordered form simply by replacing the raising and lowering operators $\hat{a}_x^\dagger$, $\hat{a}_x$, $\hat{d}_x^\dagger$, $\hat{d}_x$, $\hat{b}_x^\dagger$, and $\hat{b}_x$, by ϕ^*, ϕ, η^*, η, ξ^*, and ξ, respectively, where ϕ, η, and ξ are now complex functions of space and time. To obtain a simpler form of the action, we make a shift of variables: $\bar{\phi}=\phi^*-1$, $\bar{\eta}=\eta^*-1$, and $\bar{\xi}=\xi^*-1$, and obtain

$$\begin{aligned}S_0=\int d^dxdt[\bar{\phi}(\partial_t-D_0\nabla^2)\phi+\bar{\xi}(\partial_t-D_0\nabla^2)\xi\\+\bar{\eta}(\partial_t-D_0\nabla^2)\eta-g\bar{\xi}\bar{\phi}\xi-g\bar{\phi}\xi+k\bar{\phi}\phi\\-(\bar{\phi}\bar{\eta}+\bar{\phi}+\bar{\eta}-\bar{\xi})(f\xi-h\phi\eta)],\end{aligned} \tag{13}$$

where d is the spatial dimensionality of the system. Motivated by the fact that the model possess a quantity conserved globally, $C_B+C_D=\rho$, we perform a change of variables to $\tau\equiv\eta+\xi-\rho$. This new variable corresponds to the difference between the value of the conserved quantity at each point in space and its average value over the entire lattice. To maintain the diagonal structure of the kinetic terms, we also choose $\bar{\tau}\equiv\bar{\eta}$, $\psi\equiv\xi$, and $\bar{\psi}\equiv\bar{\xi}-\bar{\tau}$. The resulting expression for the potential part of the action is

$$g(\bar{\psi}+\bar{\tau})\bar{\phi}\psi+g\bar{\phi}\psi-k\bar{\phi}\phi+(\bar{\phi}\bar{\tau}+\bar{\phi}-\bar{\psi}) \times\left(f\psi-\left(\frac{fk}{g}+h\Delta\rho\right)\phi-h\phi\tau+h\phi\psi\right). \quad (14)$$

The quartic terms are products of two unbarred fields with two barred ones; as such, in the diagrammatic analysis below, they would give rise to loops with mass raised to a positive power above $d=2$, so they are irrelevant. As we will shortly see, the critical dimension of the model is $d=4$, so we drop these terms from the subsequent analysis. Note that there are no quadratic terms involving τ so that we can diagonalize the quadratic part of the action without considering the τ field.

To perform this diagonalization we consider the quadratic part of the potential, which depends only on the remaining two fields and thus can be written as

$$(\bar{\phi} \quad \bar{\psi})\begin{pmatrix} -k-\frac{fk}{g}-h\Delta\rho & g+f \\ \frac{fk}{g}+h\Delta\rho & -f \end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}. \quad (15)$$

The diagonalization can accomplished by a linear transformation,

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}\to \mathbf{U}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \text{ and } (\bar{\phi} \quad \bar{\psi})\to(\bar{\phi} \quad \bar{\psi})\mathbf{U}^{-1}, \quad (16)$$

where $\mathbf{U}$ is the matrix whose columns are the eigenvectors of the matrix in Eq. (15). When we apply the transformation $\mathbf{U}$, the quadratic terms become diagonal and each cubic term in the action can be written as a group of terms which factorize in a convenient fashion,

$$\begin{aligned} S_d = \int d^dx dt\Bigg[& \bar{\psi}(\partial_t - D_0\nabla^2)\psi + \bar{\phi}(\partial_t - D_0\nabla^2)\phi \\ & + \bar{\tau}(\partial_t - D_0\nabla^2)\tau + D_0 m^2\bar{\psi}\psi + D_0 m_c\bar{\phi}\phi \\ & + (\bar{\phi}-\chi\bar{\psi})\left(\frac{\tilde{\lambda}}{2}\phi^2+\tilde{\delta}\phi\psi-\frac{\tilde{\gamma}}{2}\psi^2\right) \\ & - \left(\frac{\lambda}{2}\bar{\phi}^2+\delta\bar{\phi}\bar{\psi}-\frac{\gamma}{2}\bar{\psi}^2\right)(\phi-\kappa\psi) \\ & + (\bar{\phi}-\chi\bar{\psi})(\tilde{\mu}\phi\tau+\tilde{\nu}\psi\tau)-(\mu\bar{\phi}\bar{\tau}+\nu\bar{\psi}\bar{\tau})(\phi-\kappa_1\psi)\Bigg]. \end{aligned} \quad (17)$$

Here m, m_c, λ, $\tilde{\lambda}$, δ, $\tilde{\delta}$, γ, $\tilde{\gamma}$, χ, κ, and κ_1 are coupling constants which depend on the original rate constants f, g, h, and k. Since only the structure of the Lagrangian, not the specific values of the coupling constants, plays a role in the field theoretic treatment of this model, the precise expressions relating the two sets of constants are not needed. The coefficient of the $\bar{\psi}\psi$ term is written as m^2 to emphasize the fact that this term never changes sign and this field therefore does not display critical behavior, while the coefficient of $\bar{\phi}\phi$ is written as m_c to indicate that this parameter goes to zero at the critical point.

Since the action is dimensionless, the Lagrangian must have the dimensions $[L]^{-d}[T]^{-1}$ in spatial dimension d. We work in units of time in which the diffusion constant is dimensionless, so $[T]=[L]^2$. Examining the derivative terms in the action shows that the products of barred and unbarred fields such as $\bar{\phi}\phi$ must have dimension $[L]^{-d}$. To choose the dimension of the individual fields, note that the form of the action is unchanged under the transformation,

$$t\to -t; \quad \phi\leftrightarrow -\bar{\phi}; \quad \psi\leftrightarrow -\bar{\psi}; \quad \tau\leftrightarrow -\bar{\tau}. \quad (18)$$

In other words, the action possesses a symmetry corresponding to time reversal and exchanging barred and unbarred fields. Due to this symmetry we choose the dimension of the barred and unbarred fields to be the same ($[L]^{-d/2}$). It follows from this that the dimensions of the coupling constants are: $[\tilde{\lambda}]=[\lambda]=[\tilde{\delta}]=[\delta]=[\tilde{\gamma}]=[\gamma]=[L]^{(4-d)/2}$ which means that these terms are relevant at $d=4$ and below. In summary, the critical dimension can be deduced from the form of the terms in the diagrammatic perturbative expansion (of the full action), and the symmetry in Eq. (18) suggests that all conjugate pairs of fields should be rescaled equally.

III. RENORMALIZATION

A. Large mass reduction

To proceed it is helpful to notice that the quadratic ("mass") terms for each of the fields in the above action [Eq. (17)] are qualitatively different. The ϕ field exhibits critical behavior in that its mass goes to zero at a specific point. In contrast, the ψ field has a mass term which is positive at all times, and the τ field has no mass term. We therefore expect the τ field to play the role of a background diffusive field and the ψ field to relax exponentially at all parameter values. Due to this exponential relaxation, it is reasonable to expect the effects of ψ to be suppressed in the long time limit in which we are interested. Specifically,

$$G^{(n)}_{full}(\omega,p)=Z(m)G^{(n)}_{eff}(\omega,p)[1+O(1/m)], \quad (19)$$

where m is the mass of the heavy field and $G^{(n)}_{eff}$ has no dependence on m. In essence, as discussed below and in the Appendix, the heavy field ψ can be dropped from the model by proper rescaling and introduction of vertices that might arise from diagrams containing heavy propagators. That is, if there are vertices which are not present in the Lagrangian, but which would be capable of receiving perturbative corrections containing the heavy fields, such vertices must be introduced by hand into the effective Lagrangian which results from removing the heavy field. However, it can be seen that there are no such new vertices in our model (i.e., the Lagrangian is complete).

To understand what it means for a Lagrangian to be complete, it is helpful to consider what terms could be added to

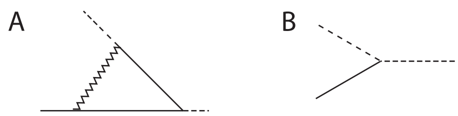

FIG. 1. Illustration of a Lagrangian which is not complete. (a) Inclusion of the term $\bar{\phi}^2\tau$ in the full Lagrangian [Eq. (17)] would enable the construction of this diagram. Solid, dashed, and wiggly lines represent the propagators for the ϕ, τ, and ψ fields, respectively. In these diagrams, time advances from left to right. (b) The diagram in (a) is a correction to this vertex which would have to be introduced by hand into the Lagrangian when performing the large mass reduction.

our Lagrangian to render it incomplete. As one example, consider the Lagrangian of Eq. (17) with an additional term $\theta\bar{\phi}^2\tau$, where θ is a new coupling constant. Then it would be possible to combine this vertex with the vertices $\tilde{\delta}\bar{\phi}\phi\psi$ and $\nu\bar{\psi}\bar{\tau}\phi$ to create the diagram shown in Fig. 1(A) for the perturbation theory below. This diagram would function as a correction to a vertex $\bar{\phi}\bar{\tau}\tau$ [Fig. 1(B)] which is not present in the Lagrangian of the model. In performing the large mass reduction, we would be required to introduce this latter term by hand when removing the field ψ. The physical reason for this is that, in general, diagrams with the massive field are suppressed by a factor of $1/m^2$ and therefore need not be considered in the long time limit. However, when the vertex to which a loop diagram would serve as a correction is not present in the Lagrangian, then the loop diagram is the lowest order term in the series for the correlation function involving its external legs and therefore must be considered. In the hypothetical example above, the loop diagram in Fig. 1(A) would be the lowest order term which contributes to $\langle\bar{\phi}\bar{\tau}\tau\rangle$.

Since the Lagrangian is complete, we can remove the massive field (ψ) from the model and the only effect is to introduce an overall scale factor [$Z(m)$]. However, because we are not interested in the values of the coupling constants in the action but only their final values reached by the renormalization-group flow, the overall scale factor has no effect. Thus, we simply remove any vertices containing ψ or $\bar{\psi}$ (see the Appendix for further discussion). This leaves us with the following effective action:

$$S_{eff} = \int d^dx dt\left[\bar{\tau}(\partial_t - D_0\nabla^2)\tau + \bar{\phi}(\partial_t - D_0\nabla^2)\phi + D_0 m_c\bar{\phi}\phi + \bar{\phi}\left(\frac{\tilde{\lambda}}{2}\phi^2 + \tilde{\delta}\phi\tau\right) - \left(\frac{\lambda}{2}\bar{\phi}^2 + \delta\bar{\phi}\bar{\tau}\right)\phi\right]. \quad (20)$$

This action will serve as the basis for the field theoretic treatment in this section.

B. Diagrammatic perturbation theory

In the perturbative treatment of the reduced action, our starting point is the free theory which contains only the quadratic terms,

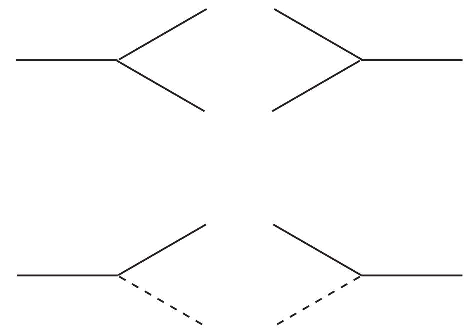

FIG. 2. Vertices used in the renormalization-group calculation. Lines correspond to the same fields as in Fig. 1.

$$S_{quad} = \int d^dx dt[\bar{\tau}(\partial_t - D_0\nabla^2)\tau + \bar{\phi}(\partial_t - D_0\nabla^2)\phi + D_0 m_c\bar{\phi}\phi]. \quad (21)$$

The two-point correlation functions ("propagators") in momentum space from the free action are

$$\langle\bar{\phi}(\mathbf{p},\omega)\phi(\mathbf{p}',\omega')\rangle = [-i\omega + D_\phi(\mathbf{p}^2 + m_c)]^{-1}\delta(\mathbf{p}+\mathbf{p}')\delta(\omega+\omega'),$$

$$\langle\bar{\tau}(\mathbf{p},\omega)\tau(\mathbf{p}',\omega')\rangle = (-i\omega + D_\tau\mathbf{p}^2)^{-1}\delta(\mathbf{p}+\mathbf{p}')\delta(\omega+\omega'). \quad (22)$$

Although the physical diffusion constants are equal, we have introduced different diffusion constants for the ϕ and τ fields because the diffusion constant for the ϕ field gets renormalized whereas that for the τ field does not, as will be shown explicitly below. Fourier transforming from ω back to t, we find that these propagators are only nonzero if they connect $\bar{\phi}$ at an earlier time with ϕ at a later time. In the diagrammatic perturbation theory below, we will represent the free propagators for ϕ and τ by solid and dashed lines, respectively.

Note that, in the action [Eq. (20)], there is no quadratic $\bar{\tau}\tau$ term, so τ does not relax exponentially at criticality. In fact, there are no quadratic terms with τ at all and higher order terms with τ also contain another barred and unbarred field. As a result, one can see by examining the vertices of the model (Fig. 2) that it is not possible to form any diagrams with external legs corresponding to $\bar{\tau}$ and τ other than the bare propagator itself. That is, the expectation value $\langle\bar{\tau}\tau\rangle$ does not get perturbative corrections in the expansion. The physical reason for this is straightforward to understand. As discussed above, the τ field corresponds to the conserved quantity $C_B + C_D$. By examining the reaction scheme which defines the model [Eq. (1)], it is clear the reactions do not change this quantity so its dynamics are purely diffusive. Thus, the expression for the bare propagator [Eq. (22)] is exact to all orders.

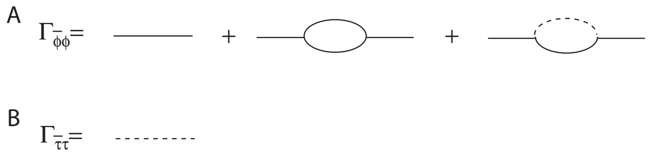

FIG. 3. Diagrammatic expansions for the two-point vertex functions to one-loop order. Note that formally the diagrammatic expressions here are actually contributions to the correlation function $G_{\bar{\phi}\phi}=\Gamma^{-1}_{\bar{\phi}\phi}$. To obtain the expression in Eq. (26), we Taylor expand $G^{-1}_{\bar{\phi}\phi}$ to first order in ϵ, which simply changes the sign of the two rightmost diagrams in this figure.

The outline of the renormalization calculation is as follows. We first note that the action [Eq. (20)] has four cubic terms and therefore the theory has four vertices (Fig. 2). We calculate correlation functions in momentum space using the standard Feynman diagram procedure [19,20]. We construct all possible diagrams from the propagators and vertices in which each field in the correlation function is represented by an external leg (Figs. 3 and 4). Each diagram corresponds to an integral in momentum space. These integrals exhibit an ultraviolet divergence as the upper limit is raised to infinity. Nevertheless, the singular part of the integral can be evaluated by performing the integral in arbitrary dimension d and analytically continuing the result to the dimension of interest. The singularities can then be removed by introducing a proper rescaling of the fields ($\bar{\phi}$ and ϕ), mass (m_c), diffusion constant (D_ϕ), and coupling constants (λ, $\tilde{\lambda}$, δ, and $\tilde{\delta}$). These rescalings, which relate the renormalized values of these quantities to their bare values, can be used to extract the dependence of these quantities on wave vector in momentum space. This, in turn, is used to examine the dependence of the n-point correlation functions on wave vector, from which the critical exponents can be extracted.

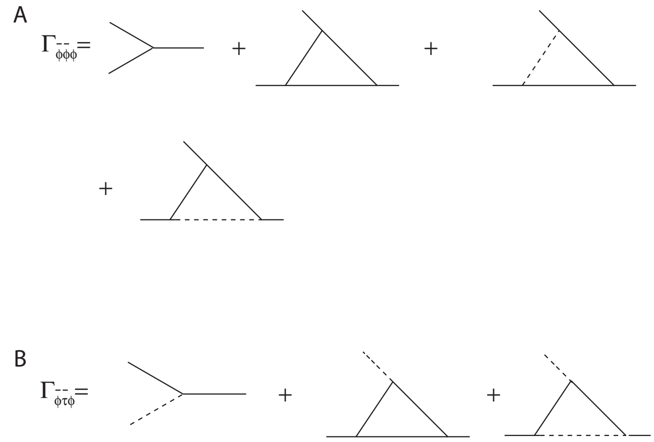

FIG. 4. Diagrammatic expansions for the three-point vertex functions to one-loop order.

We begin the renormalization calculation by evaluating the perturbative corrections to $\Gamma_{\bar{\phi}\phi}=\langle\bar{\phi}(\mathbf{p},\omega)\phi(\mathbf{p}',\omega')\rangle^{-1}$. These are given by the two diagrams in Fig. 3(A). We will describe the evaluation of the first of these diagrams in detail and will only outline the calculations for the remainder of the diagrams in the paper since they are treated analogously. We denote the spatial and temporal Fourier components of the incoming propagator by $\mathbf{k}$ and ω, respectively. We then assign the Fourier components of the upper arc to be $\mathbf{p}$ and Ω. By conservation of momentum, the Fourier components of the lower arc are $\mathbf{k}-\mathbf{p}$ and $\omega-\Omega$. Using this notation, and noting that the overall symmetry factor of this diagram is 2, which corresponds to exchange of the upper and lower arcs, we find that the integral corresponding to this diagram is

$$-2\left(\frac{\lambda\tilde{\lambda}}{4}\right)\int\int\frac{d\Omega d\mathbf{p}}{[-i\Omega+D_\phi(\mathbf{p}^2+m_c)]\{-i(\omega-\Omega)+D_\phi[(\mathbf{k}-\mathbf{p})^2+m_c]\}}. \tag{23}$$

The contour integration over Ω is easily performed and the denominator factored, yielding

$$-\frac{\lambda\tilde{\lambda}}{4D_\phi}\int\frac{d\mathbf{p}}{(\mathbf{p}^2+m_c)\{1-i\omega/[2D_\phi(\mathbf{p}^2+m_c)]+\mathbf{k}^2/[2(\mathbf{p}^2+m_c)]-\mathbf{p}\cdot\mathbf{k}/(\mathbf{p}^2+m_c)\}}. \tag{24}$$

Because we are only interested in the singular part of this integral, we can formally expand the second factor in the denominator according to $1/(1-x)=1+x+x^2+\cdots$. Since higher orders in this expansion have higher powers of $\mathbf{p}$ in the denominator, these integrals converge, and we do not need to consider them. The only integrals which are divergent at the upper critical dimension $d=4$ are those from the first order in the expansion and the second order term $\int d\mathbf{p}(\mathbf{p}\cdot\mathbf{k})^2/(\mathbf{p}^2+m_c)^2$. These integrals are all straightforward to evaluate by the technique of dimensional regularization [20]. For example, working in dimension $d=4-\epsilon$, we find

$$\int\frac{d\mathbf{p}}{(\mathbf{p}^2+m_c)}=\frac{m_c}{(4\pi)^2}\Gamma\left(-1+\frac{\epsilon}{2}\right)=-\frac{m_c}{(4\pi)^2}\left(\frac{2}{\epsilon}+O(\epsilon^0)\right). \tag{25}$$

Treating the remainder of the integrals in this fashion and performing the same procedure for the second diagram in Fig. 3(A), we find

$$\Gamma_{\bar{\phi}\phi} = -i\omega + D_\phi k^2 + D_\phi m_c + \frac{\lambda\tilde{\lambda}}{4D_\phi}G\left(-\frac{2m_c}{\epsilon} + \frac{i\omega}{D_\phi\epsilon} - \frac{k^2}{2\epsilon}\right) + \frac{\delta\tilde{\delta}}{D_\phi + D_\tau}G\left(-2\frac{D_\phi m_c}{(D_\phi + D_\tau)\epsilon} + \frac{2i\omega}{(D_\phi + D_\tau)\epsilon} - \frac{2D_\phi D_\tau k^2}{(D_\phi + D_\tau)^2\epsilon}\right) \quad (26)$$

where G is a geometric factor equal to $1/(4\pi)^2$.

We now introduce rescalings of the model quantities which are necessary to make the physical quantity $\Gamma_{\bar{\phi}\phi}$ finite. As one example, consider the terms with a factor of ω,

$$-i\omega\left(1 - \frac{\lambda\tilde{\lambda}G}{4D_\phi^2\epsilon} - \frac{2\delta\tilde{\delta}G}{(D_\phi + D_\tau)^2\epsilon}\right). \quad (27)$$

These terms are derived from the time derivative term in the Lagrangian ($\bar{\phi}\partial_t\phi$). Since there are no parameters accompanying this term, it can only be rendered finite by a rescaling of the fields. We thus define renormalized fields which are the original fields divided by the square root of the singular part of $\Gamma_{\bar{\phi}\phi}$,

$$\phi = Z_\phi^{1/2}\phi_R \quad \text{and} \quad \bar{\phi} = Z_\phi^{1/2}\bar{\phi}_R, \quad (28)$$

where Z_ϕ^{-1} is the term in parenthesis in Eq. (27). Explicitly, to first order in ϵ:

$$Z_\phi = 1 + \frac{1}{\epsilon}\left(\frac{\lambda_R\tilde{\lambda}_R}{4D_{\phi R}^2} + 2\frac{\delta_R\tilde{\delta}_R}{(D_{\phi R} + D_{\tau R})^2}\right), \quad (29)$$

where the subscript R denotes a renormalized constant. Explicit expressions for these renormalized constants in terms of the bare constants are given below. Note that below [Eq. (32)], we absorb the geometric factor G into the definition of the renormalized coupling constants, so it does not appear in any subsequent equations. Divergent terms with $\mathbf{k}^2$ are derived from the diffusion term in the action and can be made finite by introducing a renormalization of the diffusion constants. Because this term also includes the fields themselves it is necessary to include Z_ϕ as well in order to cancel the effect of the field renormalization which we just described. Thus we find

$$D_\phi = Z_\phi^{-1}Z_{D_\phi}D_{\phi R}, \quad (30)$$

where

$$Z_{D_\phi} = 1 + \frac{1}{\epsilon}\left(\frac{\lambda_R\tilde{\lambda}_R}{8D_{\phi R}^2} + 2\frac{D_{\tau R}\delta_R\tilde{\delta}_R}{(D_{\phi R} + D_{\tau R})^3}\right). \quad (31)$$

The renormalization prescription for the remainder of the constants follows in exactly the same way.

$$m_c = Z_{D_\phi}^{-1}Z_{m_c}k^{-2}m_{cR},$$

$$\tilde{\lambda} = Z_\phi^{-3/2}Z_{\tilde{\lambda}}G^{-1/2}k^{\epsilon/2}\tilde{\lambda}_R,$$

$$\tilde{\delta} = Z_\phi^{-1}Z_{\tilde{\delta}}G^{-1/2}k^{\epsilon/2}\tilde{\delta}_R, \quad (32)$$

and the rescalings for λ and δ follow from simply exchanging the tilde in the last two equations in Eq. (32). The factors of k in these expressions are included so that the renormalized quantities are dimensionless. This will be important in the analysis of the renormalization-group equation below. Z_{m_c} can be calculated from the expression for $\Gamma_{\bar{\phi}\phi}$ in a manner similar to Z_ϕ and Z_{D_ϕ},

$$Z_{m_c} = 1 + \frac{1}{\epsilon}\left(\frac{\lambda_R\tilde{\lambda}_R}{2D_{\phi R}^2} + 2\frac{\delta_R\tilde{\delta}_R}{(D_{\phi R} + D_{\tau R})^2}\right). \quad (33)$$

To compute the renormalization factors for the coupling constants, it is necessary to calculate the singularities which arise in the computation of the three-point correlation functions. It is enough to calculate the contributions to $\tilde{\lambda}$ and $\tilde{\delta}$, as the other ones follow by simply exchanging constants with and without a tilde in the resulting expressions. The necessary diagrams are shown in Fig. 4. The calculation of singularities and the renormalization of the coupling constants proceeds exactly as in the two-point case above. It is somewhat simplified because in this case we do not need to consider the diagrams with arbitrary input momentum, but rather it is sufficient to calculate them in the case $k=0$. The reason is that these terms have one more propagator than the terms for the two-point function and therefore the integrals have two more powers of the momentum variable p in the denominator. Thus, if we performed a Taylor expansion similar to the one we used to evaluate the singular part of the integral in Eq. (24), all the k-dependent terms would be finite. We find for the renormalization factors,

$$Z_{\tilde{\lambda}} = 1 + \frac{1}{\epsilon}\left(\frac{\lambda_R\tilde{\lambda}_R}{D_{\phi R}^2} + 4\frac{\delta_R\tilde{\delta}_R}{(D_{\phi R} + D_{\tau R})^2} + 2\frac{\delta_R\tilde{\delta}_R}{D_{\phi R}(D_{\phi R} + D_{\tau R})}\right),$$

$$Z_{\tilde{\delta}} = 1 + \frac{1}{\epsilon}\left(\frac{\lambda_R\tilde{\lambda}_R}{2D_{\phi R}^2} + 2\frac{\delta_R\tilde{\delta}_R}{(D_{\phi R} + D_{\tau R})^2}\right). \quad (34)$$

Note that in each expression above, the coupling constants λ and δ do not appear alone, but rather the scale factors only depend on the ratio of the coupling constants and the diffusion constants. Calculation of the renormalization-group flow for any of these quantities alone shows that each of these quantities flows to infinity. However, it is reasonable to expect the ratios which appear in the above rescalings to remain finite on physical grounds. Thus we now define $\lambda' \equiv \lambda_R/D_{\phi R}$ and similarly for the other coupling constants. Furthermore, because D_ϕ flows to infinity but D_τ does not change in the renormalization-group analysis, we can set the ratio $D_\tau/D_\phi=0$ in the analysis below. Note that this means that the second term in parenthesis in Eq. (31) can be set to zero.

We now calculate the β functions, which define how the coupling constants depend on wave vector:

$$k\frac{d\lambda'(k)}{dk} = -\beta[\lambda'(k)]. \qquad (35)$$

A similar equation holds for each coupling constant. Since for any q,

$$\frac{d\ln q}{d\ln k} = \frac{k}{q}\frac{dq}{dk} = -\frac{\beta(q)}{q}, \qquad (36)$$

these functions can be calculated according to

$$\beta(\tilde{\lambda}') = \tilde{\lambda}'\frac{\partial}{\partial \ln k}[\ln(Z_\phi^{3/2}Z_{\tilde{\lambda}}^{-1}) - \ln(Z_\phi Z_{D_\phi}^{-1})] - \tilde{\lambda}'\frac{\epsilon}{2},$$

$$\beta(\tilde{\delta}') = \tilde{\delta}'\frac{\partial}{\partial \ln k}[\ln(Z_\phi Z_{\tilde{\delta}}^{-1}) - \ln(Z_\phi Z_{D_\phi}^{-1})] - \tilde{\delta}'\frac{\epsilon}{2}. \qquad (37)$$

In practice, this calculation is performed as follows. First we substitute the expressions for the Z factors derived above [Eqs. (29), (31), (33), and (34)] into Eq. (37). We then substitute the definitions of the renormalized coupling constants in terms of the bare coupling constants [Eqs. (28), (30), and (32)]. When doing this, we set $Z_q=1$ for all q (including the other terms in Z_q would only give terms which are higher-order in ϵ). Doing this and taking the derivatives yields

$$\beta(\tilde{\lambda}') = -\frac{\epsilon}{2}\tilde{\lambda}' + \frac{3}{4}\tilde{\lambda}'^2\lambda' + 5\tilde{\lambda}'\tilde{\delta}'\delta',$$

$$\beta(\tilde{\delta}') = -\frac{\epsilon}{2}\tilde{\delta}' + \frac{3}{8}\tilde{\delta}'\tilde{\lambda}'\lambda' + 2\tilde{\delta}'^2\delta'. \qquad (38)$$

The full system of β functions consists of these equations, the equations for λ' and δ' which can be obtained by exchanging quantities with and without a tilde in the above equations, and similar equations for the critical mass (m_c), the diffusion constants, and the fields ϕ and τ. The quantities $\tilde{\delta}'\delta'$ and $\tilde{\lambda}'\lambda'$ flow to the solution of the above equations, the quantities m_c, D_ϕ, and Z_ϕ flow to infinity as will be shown explicitly below [see Eq. (41)], and the quantities D_τ and Z_τ do not get renormalized. It is the values of $\tilde{\delta}'\delta'$ and $\tilde{\lambda}'\lambda'$ which determine the values of the ζ functions at the fixed point of the renormalization-group flow and hence the values of the critical exponents [see below Eqs. (41) and (42)]. The β functions have a stable fixed point given by

$$\tilde{\lambda}'\lambda' = 0 \quad \text{and} \quad \tilde{\delta}'\delta' = \frac{\epsilon}{4}. \qquad (39)$$

Note that the full system of β functions has three additional fixed points as discussed in [13]. Only one of them is stable and corresponds to the critical behavior of the diffusive epidemic process (DEP) [13,14]. It is important to note that at the fixed point corresponding to this model, the ratio $D_\tau/D_\phi \neq 0$ in contrast to the model of catalyzed autoamplification. The DEP fixed point is inaccessible in our model because it requires that certain rate constants, which are guaranteed to be positive, flow to negative values. The remaining two fixed points, one of which is the directed percolation fixed point, are unstable.

The properties of this renormalization-group fixed point determine the long time behavior of the system. As we have only a single critical degree of freedom, the renormalization-group calculation follows by standard means given the value of the coupling constants at the fixed point [19]. The renormalization-group equation gives the dependence of the renormalized density $\langle\phi\rangle$ on changing the scale in momentum space

$$\left[k\frac{\partial}{\partial k} + \beta(m_c)\frac{\partial}{\partial m_c} + \beta(D_\phi)\frac{\partial}{\partial D_\phi} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \beta(\tilde{\lambda})\frac{\partial}{\partial \tilde{\lambda}} + \beta(\delta)\frac{\partial}{\partial \delta} + \beta(\tilde{\delta})\frac{\partial}{\partial \tilde{\delta}} + \frac{1}{2}\zeta_\phi\right]\langle\phi(k,m_c,D_\phi,\lambda,\delta,\tilde{\lambda},\tilde{\delta})\rangle = 0,$$

where $\zeta_\phi = \partial \ln Z_\phi / \partial \ln k$. The above equation is a first-order partial differential equation which can be solved by the method of characteristics. The solution is

$$\langle\phi(k,m_c,D_\phi,\lambda,\delta,\tilde{\lambda},\tilde{\delta})\rangle = |m_c|^\beta\hat{\phi}\left(\lambda^*,\tilde{\lambda}^*,\delta^*,\tilde{\delta}^*,\frac{kx}{|m_c|^{-\nu_\perp}},\frac{k^2D_\phi t}{|m_c|^{-\nu_\parallel}}\right), \qquad (40)$$

where stars indicates values at the fixed point of the renormalization-group flow. To calculate the exponents, we first calculate the flow functions

$$\zeta_\phi = \frac{\partial \ln}{\partial \ln k}Z_\phi = -\frac{\tilde{\lambda}'\lambda'}{4} - 2\tilde{\delta}'\delta',$$

$$\zeta_{D_\phi} = \frac{\partial \ln}{\partial \ln k}(Z_{D_\phi}^{-1}Z_\phi) = -\frac{\tilde{\lambda}'\lambda'}{8} - 2\tilde{\delta}'\delta',$$

$$\zeta_{m_c} = \frac{\partial \ln}{\partial \ln k}(Z_{m_c}^{-1}Z_{D_\phi}k^{-2}) = -2 + \frac{3\tilde{\lambda}'\lambda'}{8} + 2\tilde{\delta}'\delta', \qquad (41)$$

where the derivatives have been evaluated in the same way as those for calculating the β functions [Eq. (37)]. In particular, these equations show that D_ϕ flows to infinity as was assumed above. The exponents are given in terms of the fixed points of the flow functions according to

$$\nu_\perp = \frac{1}{-\zeta_{m_c}^*} = \frac{1}{2-\epsilon/2},$$

$$\nu_\parallel = \frac{2+\zeta_D^*}{-\zeta_{m_c}^*} = 1 + O(\epsilon^2),$$

$$\beta = \frac{d+\zeta_\phi^*}{-2\zeta_{m_c}^*} = 1 - \frac{\epsilon}{8} + O(\epsilon^2), \qquad (42)$$

where in the second equality in each line we have plugged in the value of the fixed point of the β functions ($\tilde{\lambda}'\lambda'=0$ and $\tilde{\delta}'\delta'=\epsilon/4$). One important feature of the model is that the result for $\nu_\perp$ is exact to all orders. This is due to a Ward identity relating $\Gamma_{\bar{\phi}\phi}$ and $\Gamma_{\bar{\phi}\bar{\tau}\phi}$ as shown in [13]. The above expressions conclude the calculation of the critical exponents of the model to one loop.

The expressions above can be compared with the mean-field critical exponents (for our model as well as DP): $\beta=1$, $\nu_\perp=1/2$, $\nu_\parallel=1$ [3]. Substituting a value of 1 for ϵ in the above expressions, one obtains $\beta=7/8$, $\nu_\perp=2/3$, and $z=\nu_\parallel/\nu_\perp=3/2$; simulations on a three-dimensional lattice yield $\beta=0.99$, $\nu_\perp=0.67$, and $z=\nu_\parallel/\nu_\perp=1.67$ [3]. Substituting a value of 2 for ϵ, one obtains $\beta=3/4$, $\nu_\perp=1$, and $z=\nu_\parallel/\nu_\perp=1$; simulations on a two-dimensional lattice yield $\beta=0.89$, $\nu_\perp=1.15$, and $z=\nu_\parallel/\nu_\perp=1.67$ [3]. Considering the order of the expansion, the agreement is quite good.

In addition, as noted above, the exponents of two well-studied models can also be derived from the β functions of our model. The exponents of the DEP [13,15] can be obtained from the full set of β functions as discussed in [13]. The exponents for DP can be derived by substituting the values of the DP fixed point ($\tilde{\lambda}'\lambda'=2\epsilon/3$ and $\tilde{\delta}'\delta'=0$) into Eq. (41) and then substituting the resulting expressions into Eq. (42). The results are

$$\nu_\perp = \frac{1}{2} + \frac{\epsilon}{16},$$

$$\nu_\parallel = 1 + \frac{\epsilon}{12},$$

$$\beta = 1 - \frac{\epsilon}{6}. \tag{43}$$

These results establish that the critical exponents of our model are different from those of the directed percolation universality class.

IV. DISCUSSION

We have examined the critical behavior of a model of catalyzed autoamplification. In this model, one species (A) binds to a catalyst (D) and the bound state (B) generates additional copies of A. This leads to a field theoretic description with three fields of which only a single field (a linear combination of C_A and C_B) displays critical behavior. A second field corresponds to a quantity whose sum over all lattice site is conserved (C_B+C_D). There are no quadratic terms in the action containing this field and therefore its value on any particular lattice site does not relax exponentially for any choice of the parameters (i.e., it is a "massless" field). The third field has quadratic terms in the action which are always negative, and therefore its average value always relaxes exponentially (i.e., it is a "massive" field). To calculate the behavior of the model at long times, we constructed a perturbation series of dimensionless quantities. The drastic differences between the fields suggested an effective action with the massive field removed and a treatment of its contributions to the β functions as a perturbation. We then employed a standard renormalization-group procedure to examine the critical behavior of the model to first order in the ϵ expansion. Subsequently, we verified that the fixed point so obtained is not changed by the presence of the massive field. This analysis established that the universality class of the model is distinct from ones studied previously.

A model very similar to the one that we examined is the diffusive epidemic process which describes population pollution [13,14]. The cubic terms in that model are the same as those that arise in the action after performing the large mass reduction [Eq. (17)] except that the coupling constants have opposite sign. The difference in sign causes the renormalization-group equations to flow to different regions of the phase space and hence different fixed points. As one example, in our model D_τ/D_ϕ converges to zero, while in the DEP the limit is finite. That the two models display different critical behavior is also supported by numerical simulations [3,15]. Another model with a conserved concentration that influences a phase transition is the conserved lattice gas [21]; however, the coupling between the ϕ and τ fields makes our model fundamentally different, as we now discuss.

At the fixed point of the renormalization-group flow, $D_\tau/D_\phi=Z_\tau/Z_\phi=0$. This means that, in this limit, the background τ field acts like a static Gaussian field without correlation between neighboring lattice sites. In this sense, it can be viewed as spatially quenched disorder. However, caution is required with respect to this interpretation because the configuration of the catalyst evolves over times that are long but still finite. Quenched disorder usually implies that the stable fixed point is shifted to either an unphysical part of the parameter space or to infinity [22]; as a result, the usual power-law scaling is not observed, and, instead, there is a slow, logarithmic dynamics [23–25]. It is possible that we recover power-law scaling in our model because the steady-state behavior of the system depends on the averaging over the different configurations of the catalyst. Because we do not determine the domain of attraction of the stable fixed point, we cannot exclude the possibility that a renormalization-group flow that was initiated from another part of the parameter space would be divergent. Investigating this possibility will be of interest in the future. Despite these caveats, our results suggest that it would be of interest to explore whether studying reaction-diffusion systems with a background species that has purely diffusive dynamics and influences the dynamics of the other species can provide a useful tool for studying the critical behavior of models with disorder. This would be of significant utility because models that exhibit disorder arise in a variety of physical contexts [26], and, in the past, it has been necessary to treat such models by introducing the disorder into the action by hand [22,23,27].

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APPENDIX: JUSTIFICATION OF THE EFFECTIVE ACTION

Our analysis in the main text relied upon a simplification of the full action Eq. (17) to the reduced action in Eq. (20).

We argued that because the coefficient of the quadratic term $\bar{\psi}\psi$ remains positive for all parameter values, this field relaxes exponentially and we can utilize a large mass expansion to remove the terms involving ψ from the Lagrangian. In this Appendix, we provide formal justification for this result. We believe that the strategy utilized here could be of general use in treating reaction-diffusion problems with both a critical and a massive field.

The minimal subtraction procedure used to treat the reduced action, while convenient, is not appropriate for models with more than one field with a mass parameter. In a model with only a single critical field, the minimal subtraction procedure can elucidate the dependence of the correlation functions on the mass yielding the critical exponents. When more than one mass scale is present, the renormalization-group equations resulting from minimal subtraction do not explicitly decouple the dependences of the correlation functions on these masses. Thus, to treat the full model, we must employ a renormalization scheme that explicitly involves the masses σ and m for the ϕ and ψ fields, respectively. While it is in principle possible to treat the full model with such a renormalization scheme, it is not practical owing to the large number of diagrams involved. As an alternative, here we demonstrate that if an analysis which ignores the contributions from the heavy field yields β functions with a certain fixed point, this fixed point will be stable in the full model as well. Contributions from the heavy field only affect the transient renormalization-group flow.

To evaluate the critical behavior of the theory, we need to compute the β functions for dimensionless combinations of the coupling and diffusion constants. Let us denote an arbitrary such ratio by α_i and all such quantities by the vector form $\boldsymbol{\alpha}$. Implicitly, we can write an equation for $\boldsymbol{\alpha}$ of the form

$$\boldsymbol{\alpha} = \mathbf{f}\left(\boldsymbol{\alpha}, \frac{\sigma}{m}, \frac{D_\phi}{D_\psi}\right). \tag{A1}$$

Differentiating this expression with respect to $\ln\sigma$ preserves the fact that the masses and diffusion constants enter only through the ratios σ/m and D_ϕ/D_ψ. The resulting expression is the β function for $\boldsymbol{\alpha}$ which can be obtained in perturbation theory by considering loop diagrams with a $\bar{\phi}\phi$ field inserted. We can divide these contributions into two groups

$$\frac{\partial}{\partial \ln \sigma}\boldsymbol{\alpha} = \boldsymbol{\beta}(\boldsymbol{\alpha}) + \frac{\sigma}{m} \cdot \boldsymbol{\beta}_m\left(\boldsymbol{\alpha}, \frac{\sigma}{m}, \frac{D_\phi}{D_\psi}\right). \tag{A2}$$

The first term on the right side is due to loops containing only the critical field ϕ, while the second term consists of contributions from the massive field ψ. Consider a fixed point of a model which ignores the heavy diagrams such that $\beta(\alpha)=0$ and $\sigma/m=0$. It is clear that

$$\boldsymbol{\beta}_m\left(\boldsymbol{\alpha}, \mathbf{0}, \frac{D_\phi}{D_\psi}\right) \tag{A3}$$

is well-defined and analytical around such finite points. That is, if the D_ϕ/D_ψ argument remains finite, $\boldsymbol{\beta}_m$ remains finite as well. Since $\boldsymbol{\beta}_m$ is multiplied by a term that goes to zero at the fixed point, the renormalization-group flow is controlled only by the $\boldsymbol{\beta}$ term in Eq. (A2). Unfortunately, it is not clear *a priori* that $\boldsymbol{\beta}_m$ is indeed finite, and we cannot simply discard the heavy loops. Specifically, there are coupling constants that only receive heavy loop corrections; when participating in light loops, they are renormalized by Z_ϕ and flow to infinity. Thus, more care is needed in dropping the $\boldsymbol{\beta}_m$ term.

To one-loop order, the $\boldsymbol{\beta}_m$ is just a polynomial in quantities which (potentially) flow to infinite values. The power at which they diverge is $O(\epsilon)$. Thus for small enough ϵ, the factor σ/m dominates the second term in Eq. (A2). However, in general such an argument cannot be extended to all dimensions. One possibility is that Lebowitz type inequalities [28,29] on the correlation functions could put bounds on the power of divergence of $\boldsymbol{\beta}_m$. Unfortunately, we are not aware of the existence of such inequalities for reaction-diffusion problems.

Instead, we employ an alternate strategy: to treat the $\boldsymbol{\beta}_m$ term as a perturbation. To this end, we first consider the flow of $\boldsymbol{\alpha}$ in Eq. (A2) with the $\boldsymbol{\beta}_m$ term removed. We then calculate the power of divergence of the different parameters that make up $\boldsymbol{\beta}_m$ to first loop order and verify that $\sigma/m \cdot \boldsymbol{\beta}_m \to 0$. A justification of the reduced action which considers higher orders is beyond the scope of the current paper.

We already know that the coupling constants of the reduced action with only light fields have a fixed point given in Eq. (39). We now consider the coupling constants which multiply diagrams involving the heavy fields. Consider first $\kappa\lambda$ and $\kappa_1\mu$ together with their symmetric counterparts $\chi\tilde{\lambda}$ and $\chi\tilde{\mu}$. A convenient rescaling factor for these constants is $\sigma^{-\epsilon/2}Z_\phi Z_\psi^{1/2}/D_\phi$ and $\sigma^{-\epsilon/2}Z_\phi^{1/2}Z_\psi^{1/2}/D_\phi$, respectively. In other words, the rescaling factor for $\kappa\lambda$ is the rescaling factor for λ multiplied by $Z_\phi^{-\frac{1}{2}}Z_\psi^{\frac{1}{2}}$. Since $Z_\phi \to \infty$, this coupling constant converges to zero. In concrete terms, the β functions for λ and μ can be written as

$$\frac{\beta(\lambda)}{\lambda} = -\frac{\epsilon}{2} + \bar{\beta}(\lambda) - \frac{3}{2}\bar{\zeta}(\lambda,\mu) + \bar{d}(\lambda,\mu),$$

$$\frac{\beta(\mu)}{\lambda} = -\frac{\epsilon}{2} + \bar{\beta}(\mu) - \bar{\zeta}(\lambda,\mu) + \bar{d}(\lambda,\mu), \tag{A4}$$

where $\bar{\beta}(\lambda)$, $\bar{\beta}(\mu)$ represent the loop contributions to the coupling constants, $\bar{\zeta}(\lambda,\mu)$ represents contributions to Z_ϕ, and $\bar{d}(\lambda,\mu)$ represents contributions to D_ϕ. Now consider the β functions for $\kappa\lambda$ and $\kappa_1\mu$,

$$\begin{aligned}\frac{\beta(\kappa\lambda)}{\kappa\lambda} &= -\frac{\epsilon}{2} + \bar{\beta}(\lambda) - \bar{\zeta}(\lambda,\mu) + \bar{d}(\lambda,\mu) - c_1(\kappa\lambda)(\chi\tilde{\lambda}) \\ &\quad - c_2(\kappa_1\mu)(\chi\tilde{\mu}) \\ &= +\frac{1}{2}\bar{\zeta}(\lambda,\mu) - c_1(\kappa\lambda)(\chi\tilde{\lambda}) - c_2(\kappa_1\mu)(\chi\tilde{\mu}),\end{aligned}$$

$$\frac{\beta(\kappa_1\mu)}{\kappa_1\mu} = -\frac{\epsilon}{2} + \bar{\beta}(\mu) - \frac{1}{2}\bar{\zeta}(\lambda,\mu) + \bar{d}(\lambda,\mu) - c_1(\kappa\lambda)(\chi\tilde{\lambda}) - c_2(\kappa_1\mu)(\chi\tilde{\mu}) = +\frac{1}{2}\bar{\zeta}(\lambda,\mu) - c_1(\kappa\lambda)(\chi\tilde{\lambda}) - c_2(\kappa_1\mu)(\chi\tilde{\mu}), \quad \text{(A5)}$$

where c_1 and c_2 are constants coming from the perturbative expansion of Z_ψ and the second equality of either equation made use of the value of $\bar{\beta}(\lambda)$ and $\bar{\beta}(\mu)$ at the fixed point [Eq. (A4)]. Since at the fixed point $\bar{\zeta}(\lambda,\mu)>0$ and the terms on the right hand side involving $\kappa\lambda$ and $\kappa_1\mu$ are zero, we have $\kappa\lambda,\ \kappa_1\mu\to 0$ although the domain of attraction for this fixed point may be finite. The coupling constants $\kappa\delta$ and $\chi\tilde{\delta}$ can be treated using an analogous argument.

The rest of the coupling constants cannot be treated this way, because there are no light loop corrections to their values. However, inspection of the diagrams corresponding to these coupling constants reveals that they each contain at most a single light field. Those without light fields do not factor into the analysis because they yield zero when differentiated in Eq. (A2). Therefore we can restrict attention to diagrams with precisely one light field. The proper rescaling of these coupling constants is $Z_\phi^{1/2}/D_\phi$. However, at the fixed point we described in the main text, $D_\phi \sim Z_\phi$ and $Z_\phi^{1/2}/D_\phi \to 0$. Furthermore, if the diagrams in question contain some of the coupling constants considered earlier, the diagram values are further suppressed by a factor of $\sigma^{\epsilon/2}$. Therefore, all of the contributions to the second term of Eq. (A2) are suppressed.

In summary, we have shown that if the renormalization-group flow with the heavy fields removed has a fixed point, such a fixed point is present in the full model as well and the flow in its neighborhood is independent of the heavy field. This allows us to drop the heavy field from consideration and justifies the analysis in the main text of the paper.

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